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On the decomposition of the modified Kadometsev–Petviashvili equation and the connection between the KN equation and a completely integrable finite-dimensional Hamiltonian system

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Abstract
The (2+1)-dimensional modified Kadomtsev–Petviashvili (mKP) equation is decomposed into the known (1+1)-dimensional Kaup–Newell (KN) equation. By using the nonlinearization of the Lax pair, a classically integrable Hamiltonian system in the Liouville sense and the involutive solution of the mKP equation (1.1) are obtained from the first two nontrivial KN equations.

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1. Introduction

More and more (2+1)-dimensional soliton equations have been decomposed into known (1+1)-dimensional soliton equation [1–5]. For example, the KP and mKP equations were decomposed into the AKNS, Chen-lee-liu and Jaulent–Miodek equations, and the special (2+1)-dimensional Toda equation was decomposed into the (1+1)-dimensional Toda equation.

In the present paper, by using the known (1+1)-dimensional KN equation [6], with the transformation $w = \frac{1}{2}uv$ we are going to decompose the (2+1)-dimensional integrable mKP equation

$$w_t = \frac{1}{16}(w_{xxx} - 6w^2 w_x - 12 w_x \partial^{-1} w_y + 12\partial^{-1} w_{yy}) \tag{1.1}$$

where $\partial^{-1} f(x, y, t) = \int_{-\infty}^{x} f(s, y, t)\,\mathrm{d}s$. By using the nonlinearization of Lax pairs, under the constrained condition induced by the eigenfunction expression of the potential $u(x) = -\sum_{j=1}^{N} \lambda_j q_j^2(x)$, $v(x) = \sum_{j=1}^{N} \lambda_j p_j^2(x)$, we obtain a classically integrable Hamilton system in the Liouville sense [7] and an involutive solution of the mKP equation (1.1). In [3], a similar equation to (1.1) was decomposed into other (1+1)-dimensional equations, and a quasi-periodic solution was obtained.

2. The decomposition of the mKP equation

In this section we decompose the mKP equation (1.1) into the two coupled (1+1)-dimensional KN equations. To achieve this we firstly derive the KN equation hierarchy and Lax pairs. The KN equation hierarchy is the isospectral class of the eigenvalue problem:

$$\varphi_x = M\varphi \qquad M = \lambda\begin{pmatrix} -\lambda & u \\ v & \lambda \end{pmatrix} \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{2.1}$$

On the basis of investigating the zero-curvature form of the KN equation:

$$M_t - V_x + [M, V] = 0 \tag{2.2}$$

we have the fundamental identity:

$$V_x + [M, V] = M_*(P(K - \lambda^2 J)\gamma) \tag{2.3}$$

where $[M, V] = MV - VM$ is the commutator, P maps $(a, b, c)^T \to (a, b)^T$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$,

$$M_*(\delta u, \delta v)^T = \lambda\begin{pmatrix} 0 & \delta u \\ \delta v & 0 \end{pmatrix} \qquad V = \sigma(\gamma) = \begin{pmatrix} \lambda\gamma_3 & \gamma_2 \\ \gamma_1 & -\lambda\gamma_3 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & \partial & 0 \\ \partial & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J = 2\begin{pmatrix} 0 & -1 & -u \\ 1 & 0 & v \\ u & -v & \partial \end{pmatrix} \qquad \partial = \frac{\partial}{\partial x} \qquad \partial\partial^{-1} = \partial^{-1}\partial = 1.$$

The Lenard gradients g_j, the KN vector fields X_j and the jth-order KN equations are defined recursively by

$$Kg_{j-1} = Jg_j \qquad Jg_{-1} = 0 \qquad g_{-1} = (v, u, -1)^T$$
$$X_j = PJg_j \qquad \frac{\mathrm{d}}{\mathrm{d}t_j}\begin{pmatrix} u \\ v \end{pmatrix} = X_j \qquad j = 1, 2, \ldots.$$

The first few above are as follows:

$$g_{-1} = \begin{pmatrix} v \\ u \\ -1 \end{pmatrix} \qquad g_0 = \frac{1}{2}\begin{pmatrix} v_x - uv^2 \\ -u_x - u^2 v \\ uv \end{pmatrix} \qquad g_1 = \frac{1}{4}\begin{pmatrix} v_{xx} - 3uvv_x + \frac{3}{2}u^2v^3 \\ u_{xx} + 3uvu_x + \frac{3}{2}u^3v^2 \\ uv_x - vu_x - \frac{3}{2}(uv)^2 \end{pmatrix} \tag{2.4}$$

$$X_0 = \begin{pmatrix} u_x \\ v_x \end{pmatrix} \qquad X_1 = \frac{1}{2}\begin{pmatrix} -u_{xx} - (u^2v)_x \\ v_{xx} - (uv^2)_x \end{pmatrix} \qquad X_2 = \frac{1}{4}\begin{pmatrix} u_{xxx} + 3(uvu_x)_x + \frac{3}{2}(u^3v^2)_x \\ v_{xxx} - 3(uvv_x)_x + \frac{3}{2}(u^2v^3)_x \end{pmatrix}. \tag{2.5}$$

The first two nontrivial KN equations are

$$u_{t_1} = -\tfrac{1}{2}(u_{xx} + (u^2v)_x) \qquad v_{t_1} = \tfrac{1}{2}(v_{xx} - (v^2u)_x)$$
$$u_{t_2} = \tfrac{1}{4}\left(u_{xxx} + 3(uvu_x)_x + \tfrac{3}{2}(u^3v^2)_x\right)$$
$$v_{t_2} = \tfrac{1}{4}\left(v_{xxx} - 3(uvv_x)_x + \tfrac{3}{2}(u^2v^3)_x\right).$$

Let $t_1 = y, t_2 = t$. Then the first two nontrivial KN equations have the forms:

$$u_y = -\tfrac{1}{2}(u_{xx} + (u^2 v)_x) \qquad v_y = \tfrac{1}{2}(v_{xx} - (v^2 u)_x) \tag{2.6}$$

$$u_t = \tfrac{1}{4}\left(u_{xxx} + 3(uvu_x)_x + \tfrac{3}{2}(u^3 v^2)_x\right) \tag{2.7}$$

$$v_t = \tfrac{1}{4}\left(v_{xxx} - 3(uvv_x)_x + \tfrac{3}{2}(u^2 v^3)_x\right). \tag{2.8}$$

Now we consider the composition of the mKP equation (1.1). It is a well-known fact that KN equations (2.6)–(2.8) are compatible since the flows determined by them are commutable. We assume that (u, v) is a solution of equations (2.6)–(2.8), and introduces $w = \frac{1}{2}uv$. Then by direct calculation we obtain theorem 2.1.

Theorem 2.1. *Let (u, v) be a compatible solution of the KN equations (2.6)–(2.8). Then $w(x, y, t) = \frac{1}{2}u(x, y, t)v(x, y, t)$ is a solution of the (2+1)-dimensional mKP equation (1.1).*

Proof. From (2.6) we obtain

$$w_y + 3ww_x = -\tfrac{1}{4}(vu_x - uv_x)_x \qquad \partial^{-1}w_y + \tfrac{3}{2}w^2 = -\tfrac{1}{4}(vu_x - uv_x) \tag{2.9}$$

$$\partial^{-1}w_{yy} + 4ww_y - \tfrac{1}{4}w_{xxx} + 3w_x\partial^{-1}w_y + \tfrac{15}{2}w^2 w_x = -\tfrac{1}{2}(u_x v_x)_x \tag{2.10}$$

$$w_t - \tfrac{1}{4}w_{xxx} + 3ww_y + 6w^2 w_x + 3w_x\partial^{-1}w_y = -\tfrac{3}{8}(u_x v_x)_x. \tag{2.11}$$

Hence we have

$$w_t - \tfrac{1}{16}w_{xxx} + \tfrac{3}{8}w^2 w_x + \tfrac{3}{4}w_x\partial^{-1}w_y - \tfrac{3}{4}\partial^{-1}w_{yy} = 0. \tag{2.12}$$

□

Remark. Under the transformation $(x, y, t, w) \to (x, -2y, 16t, q)$, equation (1.1) is transformed into equation (1.11.28) in [8]. Hence equation (1.1) is called the mKP equation.

3. The KN–Bargmann system

Consider N copies of the KN eigenvalue problem (2.1):

$$\begin{pmatrix} q_j \\ p_j \end{pmatrix}_x = \lambda_j \begin{pmatrix} -\lambda_j & u \\ v & \lambda_j \end{pmatrix} \begin{pmatrix} q_j \\ p_j \end{pmatrix} \qquad j = 1, 2, \ldots, N \tag{3.1}$$

with distinct eigenvalues $\lambda = \lambda_j, \lambda_i \neq \lambda_j$ $(i \neq j, j = 1, 2, \ldots, N)$.

Let $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N), q = (q_1, q_2, \ldots, q_N)^T, p = (p_1, p_2, \ldots, p_N)^T$ and $\langle q, p\rangle = \sum_{j=1}^N q_j p_j$ which is the standard inner product in $\mathbf{R}^N$. We give the transformation

$$u(x) = -\langle Aq, q\rangle = -\sum_{j=1}^{N} \lambda_j q_j^2(x) \qquad v(x) = \langle Ap, p\rangle = \sum_{j=1}^{N} \lambda_j p_j^2(x). \tag{3.2}$$

Then linear equation (3.1) is transformed into a system of the nonlinear equation:

$$\begin{aligned} q_x &= -A^2 q - \langle Aq, q\rangle Ap = \frac{\partial H_0}{\partial p} \\ p_x &= A^2 p + \langle Ap, p\rangle Aq = -\frac{\partial H_0}{\partial q} \\ H_0 &= -\langle A^2 q, p\rangle - \tfrac{1}{2}\langle Aq, q\rangle\langle Ap, p\rangle. \end{aligned} \tag{3.3}$$

This procedure is called nonlinearization of the Lax pairs [9–11]. To discuss the integrability of (3.3), we first give the two very useful lemmas.

Lemma 3.1. *If M and V are two smooth two-order matrices,* $\mathrm{tr}(M) = 0$ *and* $V_x = [M, V]$, *then* $F = \det V$ *is constant along the x-flow.*

Proof. Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11} \end{pmatrix} \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

From $V_x = [M, V] = MV - VM$ we have

$$\frac{\mathrm{d}V_{11}}{\mathrm{d}x} = M_{12}V_{21} - M_{21}V_{12} = -\frac{\mathrm{d}V_{22}}{\mathrm{d}x}$$
$$\frac{\mathrm{d}V_{12}}{\mathrm{d}x} = -2M_{11}V_{12} + M_{12}V_{22} - V_{11}M_{12}$$
$$\frac{\mathrm{d}V_{21}}{\mathrm{d}x} = -2M_{11}V_{21} + M_{21}V_{11} - V_{22}M_{21}.$$

By a direct calculation we have

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(V_{11}V_{22} - V_{12}V_{21}) \\ &= V_{11}\frac{\mathrm{d}V_{22}}{\mathrm{d}x} + V_{22}\frac{\mathrm{d}V_{11}}{\mathrm{d}x} - V_{12}\frac{\mathrm{d}V_{21}}{\mathrm{d}x} - V_{21}\frac{\mathrm{d}V_{12}}{\mathrm{d}x} = 0.\end{aligned}$$

□

Lemma 3.2 (Liouville–Arnold lemma [7]). *If, in a canonical system with n degrees of freedom (i.e. with a 2n-dimensional phase space), n independent first integrals in involution are known, then the system is integrable by quadratures.*

Now we consider the problem of integrability of the KN–Bargmann system (3.3). On standardization condition that $\int_{-\infty}^{\infty} \left(vp_j^2 + 4\lambda_j q_j p_j - up_j^2\right) \mathrm{d}x = 1$, the gradient $\nabla\lambda_j = (\delta\lambda_j/\delta u, \delta\lambda_j/\delta v)^T = \left(\lambda_j p_j^2, -\lambda_j q_j^2\right)^T$. We extend $\nabla\lambda_j$ into $\nabla\lambda_j = \left(\lambda_j p_j^2, -\lambda_j q_j^2, q_j p_j\right)^T$, which satisfies the Lenard eigenvalue problem $\left(K - \lambda_j^2 J\right)\nabla\lambda_j = 0$. Condition (3.2) is put into the general form

$$g_{-1} = \sum_{j=1}^{N} \nabla\lambda_j = (\langle Ap, p\rangle, -\langle Aq, q\rangle, \langle q, p\rangle)^T. \tag{3.4}$$

Consider the Lenard eigenvalue problem

$$(K - \lambda^2 J)G_\lambda = 0. \tag{3.5}$$

The solution of (3.5) is

$$G_\lambda = \sum_{j=1}^{N} \frac{\nabla\lambda_j}{\lambda^2 - \lambda_j^2} = \sum_{j=1}^{N} \frac{1}{\lambda^2 - \lambda_j^2} \begin{pmatrix} \lambda_j p_j^2 \\ -\lambda_j q_j^2 \\ q_j p_j \end{pmatrix} = \begin{pmatrix} \Omega_\lambda(Ap, p) \\ -\Omega_\lambda(Aq, q) \\ \Omega_\lambda(q, p) \end{pmatrix} \tag{3.6}$$

where

$$\begin{aligned}\Omega_\lambda(\xi, \eta) &= \langle(\lambda^2 I - A^2)^{-1}\xi, \eta\rangle = \sum_{j=1}^{N} \frac{\xi_j \eta_j}{\lambda^2 - \lambda_j^2} \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=1}^{N} \lambda_j^{2k} \xi_j \eta_j\right) \lambda^{-2k} = \sum_{k=0}^{\infty} \langle A^{2k}\xi, \eta\rangle \lambda^{-2k}\end{aligned}$$

$$\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \qquad \eta = (\eta_1, \eta_2 \ldots, \eta_n)^T.$$

In fact

$$(K-\lambda^2 J)G_\lambda = \sum_{j=1}^{N} \frac{K\nabla\lambda_j - \lambda^2 J\nabla\lambda_j}{\lambda^2-\lambda_j^2} = \sum_{j=1}^{N} \frac{\lambda_j^2 J\nabla\lambda_j - \lambda^2 J\nabla\lambda_j}{\lambda^2-\lambda_j^2}$$
$$= \sum_{j=1}^{N}(-J\nabla\lambda_j) = J\left(-\sum_{j=1}^{N}\nabla\lambda_j\right) = -Jg_{-1} = 0. \tag{3.7}$$

By the fundamental identities (2.3) and (3.7), the Lax equation along the x-flow

$$V_x = [M, V]$$

has a solution

$$V_\lambda = \sigma(G_\lambda) = \begin{pmatrix} \lambda\Omega_\lambda(q,p) & -\Omega_\lambda(Aq,q) \\ \Omega_\lambda(Ap,p) & -\lambda\Omega_\lambda(q,p) \end{pmatrix} \tag{3.8}$$

which is called the Lax matrix of the KN–Bargmann system (3.3). By using lemma 3.1, (2.3) and (3.5) we obtain that $F_\lambda = \det V_\lambda$ is constant along the x-flow. Therefore we have the generating function of integrals of (3.3):

$$F_\lambda = -\lambda^2\Omega_\lambda^2(q,p) + \Omega_\lambda(Aq,q)\Omega_\lambda(Ap,p) = \sum_{k=0}^{\infty} F_{k-1}\lambda^{-2(k+1)} \tag{3.9}$$

where

$$\begin{aligned} &F_{-1} = -\langle q,p\rangle^2 = -1 \qquad \langle q,p\rangle = -1 \\ &F_0 = \langle Aq,q\rangle\langle Ap,p\rangle + 2\langle A^2q,p\rangle \\ &F_k = \sum_{m+n=k}\langle A^{2m+1}q,q\rangle\langle A^{2n+1}p,p\rangle - \sum_{m+n=k+1}\langle A^{2m}q,p\rangle\langle A^{2n}q,p\rangle \qquad k=0,1,2,\ldots \\ &H_0 = -\tfrac{1}{2}F_0. \end{aligned} \tag{3.10}$$

By comparing the coefficients of the $\lambda^{-(2k+1)}$ in (3.9), F_{-1}, F_0 and F_k are obtained.

We consider the generating function F_λ as a Hamiltonian in the symplectic space $(\mathbf{R}^{2N}, \mathrm{d}p\wedge\mathrm{d}q)$. The canonical equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} q_k \\ p_k \end{pmatrix} = \begin{pmatrix} \partial F_\lambda/\partial p_k \\ -\partial F_\lambda/\partial q_k \end{pmatrix} \qquad k=1,2,\ldots,N.$$

By a direct calculation we obtain

$$\begin{aligned} \partial F_\lambda/\partial p_k &= -2\lambda^2\Omega_\lambda(q,p)\left(\lambda^2-\lambda_k^2\right)^{-1}q_k + 2\Omega_\lambda(Aq,q)\left(\lambda^2-\lambda_k^2\right)^{-1}Ap_k \\ &= -2\lambda V_\lambda^{11}\left(\lambda^2-\lambda_k^2\right)^{-1}q_k - 2V_\lambda^{12}\left(\lambda^2-\lambda_k^2\right)^{-1}Ap_k \\ \partial F_\lambda/\partial q_k &= -2\lambda^2\Omega_\lambda(q,p)\left(\lambda^2-\lambda_k^2\right)^{-1}p_k + 2\Omega_\lambda(Ap,p)\left(\lambda^2-\lambda_k^2\right)^{-1}Aq_k \\ &= -2\lambda V_\lambda^{11}\left(\lambda^2-\lambda_k^2\right)^{-1}p_k + 2V_\lambda^{21}\left(\lambda^2-\lambda_k^2\right)^{-1}Aq_k. \end{aligned}$$

Hence we get the canonical equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} q_k \\ p_k \end{pmatrix} = W_\lambda(\lambda,\lambda_k)\begin{pmatrix} q_k \\ p_k \end{pmatrix} \qquad k=1,\ldots,N \tag{3.11}$$

where

$$W_\lambda(\lambda,\mu) = -\frac{2\lambda}{\lambda^2-\mu^2}V_\lambda + \frac{2}{\lambda+\mu}\begin{pmatrix} 0 & V_\lambda^{12} \\ V_\lambda^{21} & 0 \end{pmatrix}. \tag{3.12}$$

Proposition 3.1. *The Lax matrix V_λ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda} V_\mu = [W(\lambda, \mu), V_\mu] \qquad \forall \lambda, \mu \in \mathbf{C} \tag{3.13}$$

$$(F_\mu, F_\lambda) = 0 \qquad \forall \lambda, \mu \in \mathbf{C} \tag{3.14}$$

$$(F_j, F_k) = 0 \qquad \forall j, k = 0, 1, 2, \ldots \tag{3.15}$$

where $(\cdot, \cdot)$ is the Poisson bracket in $(\mathbf{R}^{2N}, \mathrm{d}p \wedge \mathrm{d}q)$.

Proof. By a direct calculation we get

$$\begin{aligned}
\frac{\mathrm{d}V_\mu^{11}}{\mathrm{d}\tau_\lambda} &= \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(\mu \Omega_\mu(q, p)) \\
&= \mu \left(\left\langle (I\mu^2 - A^2)^{-1} \frac{\mathrm{d}q}{\mathrm{d}\tau_\lambda}, p \right\rangle + \left\langle (I\mu^2 - A^2)^{-1} q, \frac{\mathrm{d}p}{\mathrm{d}\tau_\lambda} \right\rangle \right) \\
&= \mu \left(\left\langle (I\mu^2 - A^2)^{-1} \frac{\partial F_\lambda}{\partial p}, p \right\rangle + \left\langle (I\mu^2 - A^2)^{-1} q, -\frac{\partial F_\lambda}{\partial q} \right\rangle \right) \\
&= -\frac{2}{\lambda + \mu}\left(V_\lambda^{12} V_\mu^{21} - V_\mu^{12} V_\lambda^{21}\right) - \frac{2}{\lambda^2 - \mu^2}\left(V_\lambda^{12} V_\mu^{21} - V_\mu^{12} V_\lambda^{21}\right) \\
\frac{\mathrm{d}V_\mu^{12}}{\mathrm{d}\tau_\lambda} &= \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(-\Omega_\mu(Aq, q)) = -2\left\langle (I\mu^2 - A^2)^{-1} Aq, \frac{\mathrm{d}q}{\mathrm{d}\tau_\lambda} \right\rangle \\
&= -2\left\langle (I\mu^2 - A^2)^{-1} Aq, \frac{\partial F_\lambda}{\partial p} \right\rangle \\
&= \frac{4}{\lambda + \mu} V_\lambda^{12} V_\mu^{11} + \frac{4\lambda}{\lambda^2 - \mu^2}\left(V_\lambda^{12} V_\mu^{11} - V_\lambda^{11} V_\mu^{12}\right) \\
\frac{\mathrm{d}V_\mu^{21}}{\mathrm{d}\tau_\lambda} &= \frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(\Omega_\mu(Ap, p)) = 2\left\langle (I\mu^2 - A^2)^{-1} Ap, \frac{\mathrm{d}p}{\mathrm{d}\tau_\lambda} \right\rangle \\
&= -2\left\langle (I\mu^2 - A^2)^{-1} Ap, \frac{\partial F_\lambda}{\partial q} \right\rangle \\
&= -\frac{4}{\lambda + \mu} V_\lambda^{21} V_\mu^{11} + \frac{4\lambda}{\lambda^2 - \mu^2}\left(V_\lambda^{11} V_\mu^{21} - V_\lambda^{21} V_\mu^{11}\right).
\end{aligned}$$

Hence we have (3.13), which implies the invariance of $F_\mu = \det V_\mu$ along the τ_λ-flow. By the definition of the Poisson bracket we have

$$(F_\mu, F_\lambda) = \frac{\mathrm{d}F_\mu}{\mathrm{d}\tau_\lambda} = 0 \qquad \forall \lambda, \mu \in \mathbf{C}. \tag{3.16}$$

From (3.9), (3.14) and (3.16) we have (3.15). By using lemma 3.2 and (3.15) we derive that the KN–Bargmann system (3.3) is classically integrable in the Liouville sense. □

4. Other integrals $\{H_k\}$ and involutive solution

In order to establish the direct relation between finite-dimensional Hamiltonian systems and the KN vector fields X_1 and X_2, we define a new set of integrals $\{H_k\}$ by

$$
\begin{aligned}
H_0 &= -\langle A^2q, p\rangle - \tfrac{1}{2}\langle Aq, q\rangle\langle Ap, p\rangle \\
H_1 &= -\langle A^4q, p\rangle - \tfrac{1}{2}\langle Aq, q\rangle\langle A^3p, p\rangle - \tfrac{1}{2}\langle Ap, p\rangle\langle A^3q, q\rangle \\
&\qquad - \tfrac{1}{2}\langle Aq, q\rangle\langle Ap, p\rangle\langle A^2q, p\rangle - \tfrac{1}{8}\langle Aq, q\rangle^2\langle Ap, p\rangle^2 \\
H_2 &= -\langle A^6q, p\rangle + \langle A^2q, p\rangle\langle A^4q, p\rangle - \tfrac{1}{2}\langle Aq, q\rangle\langle A^5p, p\rangle \\
&\qquad - \tfrac{1}{2}\langle Ap, p\rangle\langle A^5q, q\rangle - \tfrac{1}{2}\langle A^3q, q\rangle\langle A^3p, p\rangle - H_1H_0 \\
H_k &= -\frac{1}{2}\sum_{m+n=k-1} H_mH_n + \frac{1}{2}\sum_{m+n=k+1}\langle A^{2m}q, p\rangle\langle A^{2n}q, p\rangle \\
&\qquad - \frac{1}{2}\sum_{m+n=k}\langle A^{2m+1}q, q\rangle\langle A^{2n+1}p, p\rangle \qquad k = 1, 2, \ldots
\end{aligned}
\tag{4.1}
$$

which is put in the equivalent form

$$-\lambda^2 F_\lambda = (1 + H_\lambda)^2$$

with the help of the generating function

$$H_\lambda = \sum_{k=0}^{\infty} H_k\lambda^{-2(k+1)}.$$

The involutivity of $\{H_k\}$ is based on the equality

$$(H_\mu, H_\lambda) = \frac{1}{\lambda\mu\sqrt{F_\lambda F_\mu}}(F_\mu, F_\lambda) = 0 \qquad \forall\lambda, \mu \in \mathbf{C}. \tag{4.2}$$

Theorem 4.1. *The KN–Bargmann system (3.3) has a N-involutive system* $H_k, k = 0, 1, 2, \ldots, N-1$.

Denote the variable of the H_m-flow by t_m. Then the canonical equation with Hamiltonian H_m is

$$\frac{\mathrm{d}}{\mathrm{d}t_m}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \partial H_m/\partial p \\ -\partial H_m/\partial q \end{pmatrix} \qquad m = 0, 1, 2, \ldots, N-1. \tag{4.3}$$

Theorem 4.2. *Let* $x = t_0, y = t_1, t = t_2, (q, p)^T = (q(x, y, t), p(x, y, t))^T$ *be a compatible solution of (4.3)* $(m = 0, 1, 2)$. *Then* $(u, v)^T = (-\langle Aq, q\rangle, \langle Ap, p\rangle)^T = h(q, p)$ *solves the KN equations (2.6)–(2.8):*

$$\frac{\mathrm{d}}{\mathrm{d}y}\begin{pmatrix} u \\ v \end{pmatrix} = h_*(I\nabla H_1) = X_1 \tag{4.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = h_*(I\nabla H_2) = X_2. \tag{4.5}$$

Proof. $h_*(\delta u, \delta v)^T = (-2\langle Aq, \delta q\rangle, 2\langle Ap, \delta p\rangle)^T$. By direct calculation we have

$$
\begin{aligned}
u_y &= -2\langle Aq, q_y\rangle = -2\left\langle Aq, \frac{\partial H_1}{\partial p}\right\rangle = 2\langle A^5q, q\rangle - 2u\langle A^4q, p\rangle + 2\langle A^2q, p\rangle\langle A^3q, q\rangle \\
&\qquad - uv\langle A^3q, q\rangle - 2u\langle A^2q, p\rangle^2 + u^2v\langle A^2q, p\rangle
\end{aligned}
\tag{4.6}
$$

$$u_x = -2\langle Aq, q_x\rangle = -2\left\langle Aq, \frac{\partial H_0}{\partial p}\right\rangle = 2(\langle A^3q, q\rangle - u\langle A^2q, p\rangle) \tag{4.7}$$

$$v_x = 2\langle Ap, p_x\rangle = 2\left\langle A, -\frac{\partial H_0}{\partial q}\right\rangle = 2(\langle A^3p, p\rangle + v\langle A^2q, p\rangle) \tag{4.8}$$

$$\begin{aligned} u_{xx} + (u^2v)_x = & -4\langle A^5q, q\rangle + 4u\langle A^4q, p\rangle - 4\langle A^2q, p\rangle\langle A^3q, q\rangle + 2uv\langle A^3q, q\rangle \\ & + 4u\langle A^2q, p\rangle^2 - 2u^2v\langle A^2q, p\rangle. \end{aligned} \tag{4.9}$$

From (4.6) and (4.9) we obtain the first of (4.4). The second of (4.4) is similarly obtained,

$$\begin{aligned} u_t = -2\langle Aq, q_t\rangle = & -2\left\langle Aq, \frac{\partial H_2}{\partial p}\right\rangle \\ = & \, 2\langle Aq^7q\rangle - 2u\langle A^6q, p\rangle - H_1u_x - H_0u_y \\ = & \, 2\langle A^7q, q\rangle - 2u\langle A^6q, p\rangle + 2\langle A^2q, p\rangle\langle A^5q, q\rangle - uv\langle A^5q, q\rangle \\ & - 4u\langle A^2q, p\rangle\langle A^4q, p\rangle + 2\langle A^2q, p\rangle^2\langle A^3q, q\rangle - 4uv\langle A^2q, p\rangle\langle A^3q, q\rangle \\ & - 2u\langle A^2q, p\rangle^3 + 3u^2v\langle A^2q, p\rangle^2 + u^2v\langle A^4q, p\rangle + \tfrac{3}{4}u^2v^2\langle A^3q, q\rangle \\ & - \tfrac{3}{4}u^3v^2\langle A^2q, p\rangle + 2\langle A^4q, p\rangle\langle A^3q, q\rangle - u\langle A^3q, q\rangle\langle A^3p, p\rangle \\ & + u^2\langle A^2q, p\rangle\langle A^3p, p\rangle + v\langle A^3q, q\rangle^2 \end{aligned} \tag{4.10}$$

$$\begin{aligned} u_{xxx} + 3(uvu_x)_x + \tfrac{3}{2}(u^3v^2)_x = & \, 8\langle A^7q, q\rangle - 8u\langle A^6q, p\rangle + 8\langle A^2q, p\rangle\langle A^5q, q\rangle - 4uv\langle A^5q, q\rangle \\ & - 16u\langle A^2q, p\rangle\langle A^4q, p\rangle + 8\langle A^2q, p\rangle^2\langle A^3q, q\rangle - 16uv\langle A^2q, p\rangle\langle A^3q, q\rangle \\ & - 8u\langle A^2q, p\rangle^3 + 12u^2v\langle A^2q, p\rangle^2 + 4u^2v\langle A^4q, p\rangle + 3u^2v^2\langle A^3q, q\rangle \\ & - 3u^3v^2\langle A^2q, p\rangle + 8\langle A^4q, p\rangle\langle A^3q, q\rangle - 4u\langle A^3q, q\rangle\langle A^3p, p\rangle \\ & + 4u^2\langle A^2q, p\rangle\langle A^3p, p\rangle + 4v\langle A^3q, q\rangle^2. \end{aligned} \tag{4.11}$$

From (4.10) and (4.11) we obtain the first of (4.5), the second of (4.5) is similarly obtained. □

According to theorems 4.2 and 2.1 we obtain the very important theorem 4.3.

Theorem 4.3. *Let $(q, p)^T = (q(x, y, t), p(x, y, t))^T$ be a compatible solution of the H_0-flow, H_1-flow and H_2-flow with $x = t_0$, $y = t_1$ and $t = t_2$. Then*

$$w(x, y, t) = \tfrac{1}{2}uv = -\tfrac{1}{2}\langle Aq, q\rangle\langle Ap, p\rangle \tag{4.12}$$

is a solution of the (2+1)-dimensional mKP equation (1.1).

The solution $(u, v)^T = (-\langle Aq, q\rangle, \langle Ap, p\rangle)^T$ given by (3.3) is called the involutive solution of the KN equation (2.6)–(2.8). And the solution $w(x, y, t)$ given by (4.12) is called the involutive solution of the (2+1)-dimensional mKP equation (1.1).

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